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ON THE VARIATIONAL METHOD OF DERIVATION OF EQUATIONS OF STATE FOR A MATERIAL MEDIUM AND A GRAVITATIONAL FIELD

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V. A. ZHELNOROVICH and L. I. SEDOV
(Moscow)

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The formulation of equations of state for a continuous medium and gravitational field is considered in the framework of the general theory of relativity, using the fundamental variational equation. The arbitrariness of determination of equations of state is examined in detail, and problems of determining the energy-momentum tensor for the medium and gravitational field are clarified in the case of invariant dynamic and generally physical Euler's equations. Obtained formulas and deductions are specifically defined for the gravitational field in vacuum. The developed theories make possible a proper evaluation of the essence of the various assumptions prevailing in the theory of the energy-momentum pseudotensor and in the general problem of the possible form of equations of state for physical models.

1. **The variational equation.** Let us consider the four-dimensional pseudo-Riemannian space of events with metric signature $(-, -, -, +)$ in the observer's coordinate system with variables x^i $(i = 1, 2, 3, 4)$ and in the coordinate system with variables $\xi^{i'}$ moving with the continuous medium in the space of events. The dynamic equations and the equations of state for the gravitational field and the medium are obtained within the frame-work of the general theory of relativity using the variational equation of the form

$$\delta \int\limits_{V_4} \Lambda d\tau + \delta W = 0 \tag{1.1}$$

where V_4 is an arbitrary four-dimensional volume of the event space; $d\tau$ is an invariant element of volume V_4, and δW is a functional defined below by the Lagrangian Λ

$$\Lambda = \Lambda (g_{ij}, \Gamma_{ij}^k, \partial_s\Gamma_{ij}^k, x_j^i, \mu^A, \nabla_i\mu^A, K^{\wedge C}) \tag{1.2}$$

where $x_j^i = \partial x^i / \partial \xi^j$ (functions $x^i = x^i(\xi^j)$ define the law of motion of the medium); μ^A are tensor components specified in the observer's coordinate system that define various physical fields or physical properties of the medium; ∇_i is the symbol of the covariant derivative defined in the observer's coordinate system $\partial_i = \partial/\partial x^i$ are symbols of partial derivatives with respect to variables x^i; $K^{\wedge C}$ are the nonvaried tensor components specified in the related coordinate system; g_{ij} are contravariant components of the event space tensor in the observer's coordinate system, and Γ_{ij}^k are Christoffel symbols

$$\Gamma_{ij}{}^k = {}^1/_2 g^{ks}(-\partial_s g_{ij} + \partial_i g_{sj} + \partial_j g_{si})$$

If irreversible processes are taken into consideration and, also, when the coordinate system is subjected to external effects, a certain, generally nonholonomic, functional δW^* is included in the variational equation (1.1) [2 —4]. The theory developed below is based on the assumption that $\delta W^* = 0$.

Let us determine the system of subsequently used variations of determining parameters. We denote the arbitrary function $\eta^A(x^i)$ in the varied state by $\eta'^A(x^i)$. Assuming that $\eta'^A(x^i)$ differs only slightly from $\eta^A(x^i)$, the form of variation of function $\partial\eta^A$ is defined within first order smalls by the equality

$$\partial\eta^A = \eta'^A(x^i) - \eta^A(x^i) \tag{1.3}$$

It will be readily seen that the definition (1.3) implies that

$$\partial\partial_i\eta^A = \partial_i\eta'^A - \partial_i\eta^A = \partial_i\partial\eta^A$$

Besides the variation $\partial\eta^A$ we define the variation $\delta\eta^A$

$$\delta\eta^A = \partial\eta^A + \delta x^i\nabla_i\eta^A, \quad \delta x^i = x'^i(\xi^j) - x^i(\xi^j) \tag{1.4}$$

where δx^i are variations of the law of motion of the medium and are components of the four-dimensional vector.

The above definitions imply that if parameters η^A are tensor components, then variation $\partial\eta^A$ and $\delta\eta^A$ are also tensor components of the same rank as η^A.

Using the definition of variation ∂, for the quantity $x_j{}^i$ and the Christoffel symbols we can obtain [1]

$$\partial x_j{}^i = -\delta x^s\nabla_s x_j{}^i + x_j{}^s\nabla_s\delta x^i$$
$$\partial\Gamma_{ij}{}^k = {}^1/_2 g^{ks}(-\nabla_s\partial g_{ij} + \nabla_i\partial g_{sj} + \nabla_j\partial g_{si}) =$$
$${}^1/_2[-g^{ks}\delta_i{}^n\delta_j{}^q + g^{kn}(\delta_i{}^s\delta_j{}^q + \delta_j{}^s\delta_i{}^q)]\nabla_s\partial g_{nq}$$

For components $K^{\Lambda C}$ we have $\delta K^{\Lambda C} = 0$, and for the variation of a volume element we have the formula

$$\delta d\tau = (\nabla_i\delta x^i + {}^1/_2 g^{ij}\partial g_{ij})\, d\tau$$

We also determine for functions η^A the variations δ'

$$\delta'\eta^A = \partial\eta^A + \delta x^i\,\partial_i\eta^A \tag{1.5}$$

If η^A are tensor components, the covariant derivative for η^A can be defined by an equality of the form

$$\nabla_i\eta^A = \partial_i\eta^A + F^{Aj}_{Bs}\Gamma_{ij}{}^s\eta^B \tag{1.6}$$

in which coefficients F_{Bs}^{Aj} represent the sum of various products of Kronecker symbols. Using coefficients F_{Bs}^{Aj} the variation δ' of tensor components η^A can be expressed in terms of variation δ

$$\delta'\eta^A = \delta\eta^A + \delta x^i(-\nabla_i\eta^A + \partial_i\eta^A) = \delta\eta^A - F_{Bs}^{Aj}\Gamma_{ij}{}^s\eta^B\delta x^i$$

2. The Euler equations and the functional δW. Variational of the action integral can be represented in the form

$$\delta\int_{V_4}\Lambda\, d\tau \equiv \int_{V_4}(\theta^{ij}\partial g_{ij} + M_A\partial\mu^A + F_i\delta x^i)\, d\tau - \int_{V_4}\nabla_k(P_i{}^k\delta x^i + M_A{}^k\delta\mu^A + T^{kij}\delta g_{ij} + T^{ksij}\nabla_s\delta g_{ij})\, d\tau \qquad (2.1)$$

where the following notation is used:

$$\theta^{ij} = \frac{1}{2}\left\{\Lambda g^{ij} + 2\frac{\partial\Lambda}{\partial g_{ij}} + \nabla_s\left[g^{sk}\left(\frac{\delta\Lambda}{\delta\Gamma_{ij}{}^k} - \frac{\partial\Lambda}{\partial\partial_n\Gamma_{ij}{}^k}\Gamma_{nm}^m\right) - g^{ik}\left(\frac{\delta\Lambda}{\delta\Gamma_{sj}{}^k} - \frac{\partial\Lambda}{\partial\partial_n\Gamma_{sj}{}^k}\Gamma_{nm}^m\right) - g^{jk}\left(\frac{\delta\Lambda}{\delta\Gamma_{si}{}^k} - \frac{\partial\Lambda}{\partial\partial_n\Gamma_{si}{}^k}\Gamma_{nm}^m\right)\right]\right\} \qquad (2.2)$$

$$T^{ksij} = -\frac{1}{2}\left(-g^{sn}\frac{\partial\Lambda}{\partial\partial_k\Gamma_{ij}{}^n} + g^{in}\frac{\partial\Lambda}{\partial\partial_k\Gamma_{sj}{}^n} + g^{nj}\frac{\partial\Lambda}{\partial\partial_k\Gamma_{si}{}^n}\right) + N^{ksij}$$

$$T^{kij} = \frac{1}{2}\left[g^{nk}\left(\frac{\delta\Lambda}{\delta\Gamma_{ij}{}^n} - \frac{\partial\Lambda}{\partial\partial_q\Gamma_{ij}{}^n}\Gamma_{qm}^m\right) - g^{in}\left(\frac{\delta\Lambda}{\delta\Gamma_{kj}^n} - \frac{\partial\Lambda}{\partial\partial_q\Gamma_{kj}^n}\Gamma_{qm}^m\right) - g^{jn}\left(\frac{\delta\Lambda}{\delta\Gamma_{ki}^n} - \frac{\partial\Lambda}{\partial\partial_q\Gamma_{ki}^n}\Gamma_{qm}^m\right)\right] + \nabla_s N^{ksij}$$

$$P_i{}^k = -\frac{\partial\Lambda}{\partial x_j{}^i}x_j{}^k + \frac{\partial\Lambda}{\partial\nabla_k\mu^A}\nabla_i\mu^A - \Lambda\delta_i{}^k$$

$$F_i = -\frac{\partial\Lambda}{\partial x_j{}^s}\nabla_i x_j{}^s - \nabla_s\left(x_j{}^s\frac{\partial\Lambda}{\partial x_j{}^i}\right) - \frac{\partial\Lambda}{\partial K^{\wedge C}}\nabla_i K^{\wedge C}$$

$$M_A = \frac{\partial\Lambda}{\partial\mu^A} - \nabla_i\frac{\partial\Lambda}{\partial\nabla_i\mu^A}, \quad M_A{}^k = -\frac{\partial\Lambda}{\partial\nabla_k\mu^A}$$

$$\frac{\delta\Lambda}{\delta\Gamma_{ij}{}^k} = \frac{\partial\Lambda}{\partial\Gamma_{ij}{}^k} - \partial_s\frac{\partial\Lambda}{\partial\partial_s\Gamma_{ij}{}^k} + \frac{\partial\Lambda}{\partial\nabla_s\mu^A}\frac{\partial\nabla_s\mu^A}{\partial\Gamma_{ij}{}^k}$$

In formulas (2.2) the quantities $N^{ksij} = -N^{skij}$ are arbitrary differentiable functions of any arguments and $\delta/\delta\,\Gamma_{ij}{}^k$ denotes the variational derivative. Quantities θ^{ij}, T^{kij}, and T^{ksij} are by definition symmetric with respect to indices i and j

$$\theta^{ij} = \theta^{ji}, \; T^{kij} = T^{kji}, \; T^{ksij} = T^{ksji}$$

Taking into account formula (2.1) for variations of the action integral we obtain from the variational equation (1.1) the system of Euler's equations and the functional δW

$$\theta^{ij} = 0, \; F_i = 0, \; M_A = 0 \tag{2.3}$$

$$\delta W = \int_{\Sigma} (P_i^{\,k}\delta x^i + M_A^{\,k}\delta\mu^A + T^{kij}\delta g_{ij} + T^{ksij}\nabla_s\delta g_{ij})\, n_k d\sigma \tag{2.4}$$

where n_k are components of the unit vector of the external normal to the three-dimensional surface Σ that bounds volume V_4 and $d\sigma$ is an invariant element of surface Σ.

In classical theories (such as of perfect fluid, of elastic body, and of electromagnetic field) the equations that determine the terms of the integrand in δW are equations of state. The quantities $P_i^{\,k}$ which appear in δW and are defined by formula (2.2) are similar to components of the energy-momentum tensor in the special theory of relativity. Quantities of that type were used in [1, 4 – 6] in the context of the general theory of relativity. If the Lagrangian Λ is a four-dimensional scalar, the quantities $P_i^{\,k}$ are components of a second rank tensor, but in the general case when Λ is not a scalar, $P_i^{\,k}$ do not define any tensors.

Definitions (2.2) show that when Euler's equations (2.3) are invariant, the quantities T^{kij} and T^{ksij} in δW are determined by the variational equation with some arbitrariness, owing to the presence of the arbitrary function N. The definitions (2.2) of quantities P and T in the case of invariable Euler's equations can also be varied by supplementing the Lagrangian Λ by the divergent terms of the form $\nabla_i\Omega^i$, where Ω^i are specified in the form of functions that define the medium and field parameters. Thus, if Euler's equations and the functional δW expressed in the form (2.3) and (2.4), respectively, correspond to the Lagrangian Λ, then the same Euler's equations (2.3) and functional δW of the form

$$\delta W = \int_{\Sigma} (P_i^{\prime k}\delta x^i + M_A^{\,k}\delta\mu^A + T^{\prime kij}\delta g_{ij} + T^{ksij}\nabla_s\delta g_{ij} - \delta\Omega^k)\, n_k d\sigma$$

$$T^{\prime kij} = T^{kij} - {}^1/_2\Omega^k g^{ij}, \quad P_i^{\prime k} = P_i^{\,k} + \nabla_i\Omega^k - \delta_i^k\nabla_j\Omega^j$$

correspond to Lagrangian $\Lambda + \nabla_i\Omega^i$. In these equations $P_i^{\prime k}$ are defined in conformity with [5].

The form of terms of the integrand in δW substantially depends also on variations of determining parameters in δW. The use of a particular system of variations in δW is, generally, a matter of convention, and is related to considerations of convenience and the possibility of physical interpretation of equations that define δW. The selection of variation δ defined by formula (1.4) in the expression for functional δW of the form (2.4) is based on the following reasoning. First, when variations δ are used for scalar Lagrangians, the coefficients at variations are always tensor components and, consequently are of an invariant geometric nature. Second, in the case of known classic models of media the variational equation (1.1) is the equation of energy when differentials along the world lines of points of the medium are substituted for δ -variations, and the coefficients at variations δW are defined by the usual equations of state. For real increments the transition of the basic variational equation (1.1) to the equation of energy can be used as the basis for establishing the expression for Λ and in the general case fo that for δW^*.

It should be noted that the functional δW as a whole remains unchanged when, owing to the presence of arbitrary functions N or to the use of different variations of determining parameters, the definition of terms of that functional varies. However, when passing from Lagrangian Λ to Lagrangian $\Lambda + \nabla_i \Omega^i$ the entire functional δW is changed.

3. **Invariant properties of the variational equation.** Euler's equations (2.3) and formulas (2.4) for the functional δW were obtained for an arbitrary dependence of the Lagrangian Λ on arguments appearing in (1.2). Below we establish certain properties of Euler's equations and of the quantities $P_i{}^k$ contained in δW and related to the additional requirement for the Lagrangian Λ to be a scalar.

Let us assume that the Lagrangian Λ is a four-dimensional scalar and, consequently, invariant with respect to a group of arbitrary transformations of the observer's coordinate system, and calculate the variation of the action integral

$$\delta_G \int \Lambda \, d\tau$$

for the arbitrary (small) transformation $y^i = y^i(x^j)$ of variables x^i of the observer's coordinate system. For this it is sufficient to set in formula (2.1) for the arbitrary variation of the action integral the variations δx^i, δg_{ij} and $\delta \mu^A$ as follows:

$$\delta x^i = \delta \eta^i, \quad \delta g_{ij} = -\nabla_i \delta \eta_j - \nabla_j \delta \eta_i \qquad (3.1)$$
$$\delta \mu^A = F_{Bi}^{Ak} \mu^B \nabla_k \delta \eta^i$$

where within smalls of the first order $\delta \eta^i = y^i(x^j) - x^i$ and $\delta \eta_j = g_{ij} \delta \eta^i$ and the coefficients F_{Bi}^{Ak} are the same as in formula (1.6) for the covariant derivative $\nabla_i \mu^A$. Substituting expressions (3.1) into formula (2.1) for the arbitrary variation of the action integral we obtain

$$\delta_G \int\limits_{V_4} \Lambda \, d\tau \equiv \int\limits_{V_4} [2\nabla_k \theta_i{}^k + F_i - M_A \nabla_i \mu^A - \nabla_k (M_A F_{Bi}^{Ak} \mu^B)] \, \delta \eta^i d\tau - \qquad (3.2)$$
$$\int\limits_{V_4} \nabla_k [(2\theta_i{}^k - M_A F_{Bi}^{Ak} \mu^B + P_i{}^k) \delta \eta^i +$$
$$(M_A{}^k F_{Bi}^{Aj} \mu^B - 2T^{kj}_{\cdot\cdot i}) \nabla_j \delta \eta^i - 2T^{ksij} \nabla_s \nabla_j \delta \eta_i] \, d\tau$$

Since Λ is assumed to be a scalar, we have

$$\delta_G \int\limits_{V_4} \Lambda \, d\tau \equiv 0 \qquad (3.3)$$

Taking into account formula (3.2), for the variation of the action integral at coordinate transformation with $\delta\eta^i$ equal zero on surface Σ, from (3.3) we obtain the identity

$$2\nabla_k\theta_i^{\ k} + F_i - M_A\nabla_i\mu^A - \nabla_k(M_A F^{Ak}_{Bi}\mu^B) \equiv 0$$

which is satisfied only owing to the scalar properties of Lagrangian Λ, independently of the fulfilment of Euler's equations. This identity also implies that on the assumption that Λ is a scalar the second of Eqs. (2.3) (obtained with variations δx^i) is a corollary of the remaining Euler's equations.

Carrying out differentiation in the second integral of (3.2) and collecting terms containing variations $\delta\eta^i$ and their gradients, we obtain the following formula:

$$\begin{aligned}\delta_G \int_{V_4} \Lambda\, d\tau = \int_{V_4} \Big\{\Big[& F_i - M_A \nabla_i\mu^A - \nabla_k P_i^{\ k} - \frac{2}{3} T^{ksnj}(\nabla_k R_{smij} + \nabla_s R_{kmij}) + \\ & \Big(-\frac{1}{2} M_A{}^s F^{Am}_{Bq}\mu^B + T^{sm\cdot}_{\cdots q} + \nabla_k T^{ksm\cdot}_{\cdots q}\Big) R^q_{\cdot ism}\Big]\delta\eta^i + \\ & [-2\theta_i^{\ k} + M_A F^{Ak}_{Bi}\mu^B - P_i^{\ k} + \nabla_q(-M_A{}^q F^{Ak}_{Bi}\mu^B + 2T^{qk\cdot}_{\cdot\cdot i}) + \\ & \frac{1}{3} T^{qmn\cdot}_{\cdots i}(R^k_{\cdot mnq} - R^k_{\cdot qmn}) + (T^{jknq} + T^{jnkq} + T^{kjnq}) R_{qijn} + \\ & T^{jmn\cdot}_{\cdots i} R^k_{\cdot nmj}]\nabla_k\delta\eta^i + (-M_A{}^k F^{Am}_{Bi}\mu^B + 2T^{km\cdot}_{\cdot\cdot i} + 2\nabla_q T^{qkm\cdot}_{\cdots i}) \times \\ & \nabla_{(k}\nabla_{m)}\delta\eta^i + 2T^{ksij}\nabla_{(k}\nabla_s\nabla_{i)}\delta\eta_j\Big\} d\tau \end{aligned} \qquad (3.4)$$

where the parentheses enclosing subscripts at tensor components denote symmertization relative to these subscripts, and $R^k_{\cdot isj}$ are components of the curvature tensor of the pseudo-Riemannian space of events.

Using formula (3.4) we obtain from condition (3.3) which defines scalar properties the system of identities

$$\begin{aligned} & F_i - M_A\nabla_i\mu^A - \nabla_k P_i^{\ k} - {}^2/_3 T^{ksnj}(\nabla_k R_{smij} + \nabla_s R_{kmij}) + \\ & \quad (-{}^1/_2 M_A{}^s F^{Am}_{Bq}\mu^B + T^{sm\cdot}_{\cdot\cdot q} + \nabla_k T^{ksm\cdot}_{\cdots q}) R^q_{\cdot ism} \equiv 0 \\ & -2\theta_i^{\ k} + M_A F^{Ak}_{Bi}\mu^B - P_i^{\ k} + \nabla_q(-M_A{}^q F^{Ak}_{Bi}\mu^B + 2T^{qk\cdot}_{\cdot\cdot i}) + \\ & \quad {}^1/_3 T^{qmn\cdot}_{\cdots i}(R^k_{\cdot mnq} - R^k_{\cdot qmn}) + (T^{jknq} + T^{jnkq} + T^{kjnq}) R_{qijn} + \\ & \quad T^{jmn\cdot}_{\cdots i} R^k_{\cdot nmj} \equiv 0 \\ & -F^{A(m}_{Bi} M_A{}^{k)}\mu^B + 2T^{(km)\cdot}_{\cdot\cdot i} + 2\nabla_q T^{q(km)\cdot}_{\cdots i} \equiv 0 \\ & T^{(ksi)j} \equiv 0 \end{aligned} \qquad (3.5)$$

Similar identities were considered in [7] in the case when components T^{ksij} are symmetric relative to superscripts k and s, however, without taking into account the arbitrary function N.

The first of identities (3.5) shows that in virtue of Euler's equations (2.3) the components of tensor $P_i^{\ k}$ satisfy the equation

$$\nabla_k P_i^{\ k} = -{}^2/_3 T^{ksnj}(\nabla_k R_{snij} + \nabla_s R_{knij}) + (-{}^1/_2 M_A{}^s F^{Am}_{Bq}\mu^B + T^{sm\cdot}_{\cdot\cdot q} + \nabla_k T^{ksm\cdot}_{\cdot\cdot\cdot q}) R^q_{\cdot ism} \tag{3.6}$$

while the second of these identities implies that in virtue of (2.3) components $P_i^{\ k}$ may be written as

$$P_i^{\ k} = \nabla_q(-M_A{}^q F^{Ak}_{Bi}\mu^B + 2T^{qk\cdot}_{\cdot\cdot i}) + {}^1/_3 T^{qmn\cdot}_{\cdot\cdot\cdot i}(R^k{}_{mnq} - R^k_{\cdot qmn}) + (T^{jknq} + T^{jnkq} + T^{kjnq}) R_{qijn} + T^{jmn\cdot}_{\cdot\cdot\cdot i} R^k_{\cdot nmj} \tag{3.7}$$

4. **Pseudotensors of energy-momentum.** Substituting in formula (2.4) which defines functional δW the tensor variations δ' (defined by equality (1.5)) for variations δ, we obtain

$$\delta W = \int_\Sigma (t_i^{\ k}\delta x^i + t_i^{\ ks}\partial_s \delta x^i + \theta^{kij}\delta' g_{ij} + T^{ksij}\partial_s\delta' g_{ij} + M_A{}^k \delta'\mu^A)\, n_k d\sigma \tag{4.1}$$

where $t_i^{\ k}$, $t_i^{\ ks}$, and θ^{kij} are defined by

$$t_i^{\ k} = -\frac{\partial\Lambda}{\partial x_j^{\ i}} x_j^{\ k} + \frac{\partial\Lambda}{\partial\nabla_k\mu^A}\partial_i\mu^A - \theta^{knj}\partial_i g_{nj} - T^{ksnj}\partial_i\partial_s g_{nj} - \Lambda\delta_i^{\ k} + \frac{1}{\sqrt{-g}}\partial_s(N_i^{\ ks}\sqrt{-g}), \qquad g = \det\|g_{ij}\| \tag{4.2}$$

$$t_i^{\ ks} = -T^{ksnj}\partial_i g_{nj} + N_i^{\ ks}$$

$$\theta^{kij} = T^{kij} - T^{ksmj}\Gamma^i_{sm} - T^{ksmi}\Gamma^j_{sm}$$

where $N_i^{\ ks} = -N_i^{\ sk}$ are arbitrary differentiable functions, and T^{kij} and T^{ksij} are defined by equalities (2.2). Components $t_i^{\ k}$ satisfy in virtue of Euler's equations the differential law of conservation

$$\partial_k \sqrt{-g}\, t_i^{\ k} = 0 \tag{4.3}$$

and are usually called components of the energy-momentum pseudotensor. It will be seen from (4.2) that components of the pseudotensor $\sqrt{-g}\, t_i^{\ k}$ are determined by the variational equation with an accuracy within the term $\partial_s(N_i^{\ ks}\sqrt{-g})$. Note that the law of conservation is here unrelated to the invariance of the Lagrangian. Determination of the energy-momentum pseudotensor using the invariance properties of Lagrangian Λ yields the same formula (4.2) with the same arbitrariness.

5. **Transformation of functional δW.** It is obvious that owing to the indicated arbitrariness of the definition of parameters t, θ, and T in δW it is not possible to attribute to these any physical meaning without some further special assumptions. It is, however, possible to indicate an algorithm for a unique determination of terms in δW [3–5, 8–12]. The respective transformation related

to the decomposition of gradients ∇_s of variations that determine parameters in δW in terms of components tangent and normal to surface Σ, is of the form

$$\nabla_s = (\delta_s^i - \varepsilon n_s n^i)\nabla_i + \varepsilon n_s n^i \nabla_i = \zeta_s^\alpha \nabla_\alpha^* + \varepsilon n_s D/Dn \qquad (5.1)$$

where $D/Dn = n^i\nabla_i$ is the covariant derivative with respect to the normal to surface Σ; $\varepsilon = g_{ij}n^in^j$ is the sign indicator of the normal vector modulus, and ∇_α^* is the covariant derivative on surface Σ in the coordinate system with variables u^α ($\alpha = 1, 2, 3$). Components ζ_s^α in formula (5.1) are defined by the equality $\zeta_s^\alpha = G^{\alpha\beta}g_{si}\zeta_\beta^i$ in which $G^{\alpha\beta}$ are covariant components of the first metric tensor of surface Σ in the system of coordinates u^α, and $\zeta_\beta^i = \partial x^i/\partial u^\beta$ are components of basis vectors tangent to Σ. To obtain the second part of formula (5.1) it is also necessary to use the equality

$$\delta_s^j - \varepsilon n_s n^j = \zeta_s^\alpha \zeta_\alpha^j$$

which links the normal vector components n_i with components of vectors ζ_α^i tangent to surface Σ.

In the case of reasonably smooth surfaces Σ and admissible functions and their variations by substituting in formula (2.4) for δW the expressions (5.1) for gradients ∇_s, it is possible to transform the formula for δW to the form

$$\delta W = \int_\Sigma \left(n_k P_i{}^k \delta x^i + n_k M_A{}^k \delta\mu^A + T_{(0)}^{ij}\delta g_{ij} + T_{(1)}^{ij}\frac{D}{Dn}\,\delta g_{ij}\right) d\sigma \qquad (5.2)$$

where

$$T_{(1)}^{ij} = \varepsilon n_k n_s T^{ksij}, \qquad T_{(0)}^{ij} = n_k T^{kij} - \nabla_\alpha^*\left(\zeta_s^\alpha n_k T^{ksij}\right) \qquad (5.3)$$

Using the derived equations of the theory of surfaces [15]

$$\nabla_\alpha^* \zeta_\beta^s = b_{\alpha\beta} n^s, \quad \nabla_\alpha^* n_k = -\,\varepsilon b_{\alpha\beta}\zeta_k^\beta$$

where $b_{\alpha\beta}$ are components of the second metric tensor of surface Σ; the expression for quantities $T_{(0)}^{ij}$ in (5.3) can, also, be written in the form

$$T_{(0)}^{ij} = n_k\left(T^{kij} - \nabla_s T^{ksij} + \varepsilon n_s \frac{D}{Dn} T^{ksij}\right) +$$
$$T^{ksij}\left(\varepsilon b_{\alpha\beta}\zeta_k^\alpha \zeta_s^\beta - b_\alpha^\alpha n_k n_s\right), \qquad b_\alpha^\alpha = G^{\alpha\beta}b_{\alpha\beta}$$

The quantities $P_i{}^k n_k$, $T_{(0)}^{ij}$, and $T_{(1)}^{ij}$ in (5.2) are uniquely determined for a specified Lagrangian Λ by the variational equation and are independent of the arbitrary functions N (this can be checked directly). Relationships at discontinuities and boundary conditions that correspond to the selected Lagrangian are formulated [3, 4, 8 – 12] using the above quantities determined at discontinuities and boundaries.

When $\Lambda = \Lambda(R, g_{ij}, \mu^A, \nabla_k \mu^A, x_j{}^i, K^{\wedge C})$ the formula of type (5.2) for δW was derived by Sedov [4], and for δW with noncovariant derivatives $\partial/\partial n = n^i\partial_i$ appeared in [10, 11] and in [3, 9] in the context of the special theory of relativity.

6. The classical general theory of relativity. Let us now apply the derived formulas to a gravitational field in vacuum when the Lagrangian Λ is defined by

$$\Lambda = \frac{1}{2\varkappa} R + \nabla_i \Omega^i \tag{6.1}$$

where R is the scalar curvature of the space of events, $\varkappa$ is the Einstein gravitational constant, and Ω^i are specified functions whose form will not be defined here (for possible definitions of functions Ω^i see [5]). In this case Euler's equations for the event space metric are expressed in terms of Einstein's equations

$$R_{ij} - {}^1/_2 R g_{ij} = 0$$

where $R_{ij} = R^{m}_{\cdot imj}$ are components of the Ricci tensor and the functional δW is of the form

$$\delta W = \int_\Sigma (P_i{}^k \delta x^i + T^{kij} \delta g_{ij} + T^{ksij} \nabla_s \delta g_{ij} - \delta \Omega^k) n_k d\sigma \tag{6.2}$$

where

$$\begin{aligned} P_i{}^k &= -\frac{1}{2\varkappa} R \delta_i{}^k + \nabla_i \Omega^k - \delta_i{}^k \nabla_j \Omega^j \\ T^{kij} &= -{}^1/_2 \Omega^k g^{ij} + \nabla_s N^{ksij} \\ T^{ksij} &= \frac{1}{2\varkappa} [g^{ij} g^{ks} - {}^1/_2 (g^{is} g^{jk} + g^{js} g^{ik})] + N^{ksij} \end{aligned} \tag{6.3}$$

Formula (6.2) for δW is given in [1] for $\Omega^i = 0$ and $N^{ksij} = 0$. Functions Ω^i were considered in [5].

Separating in (6.2) the derivatives with respect to the normal we can express δW in the form

$$\delta W = \int_\Sigma \left(P_i{}^k n_k \delta x^i + T^{ij}_{(0)} \delta g_{ij} + T^{ij}_{(1)} \frac{D}{Dn} \delta g_{ij} - n_k \delta \Omega^k \right) d\sigma$$

where

$$T^{ij}_{(1)} = \frac{1}{2\varkappa} (g^{ij} - \varepsilon n^i n^j)$$

$$T^{ij}_{(0)} = -\frac{1}{2} n_k \Omega^k g^{ij} + \frac{1}{2\varkappa} (b_\alpha{}^\alpha n^i n^j - \varepsilon b^{\alpha\beta} \zeta_\alpha{}^i \zeta_\beta{}^j)$$

Using the noncovariant variations $\delta' g_{ij}$ we obtain the following expression for δW:

$$\delta W = \int_\Sigma (t_i{}^k \delta x^i + t_i{}^{ks} \partial_s \delta x^i + \theta^{kij} \delta' g_{ij} + T^{ksij} \partial_s \delta' g_{ij} - \delta' \Omega^k) n_k d\sigma \tag{6.4}$$

where

$$t_i^{\ k} = \frac{1}{\varkappa}\left(R_i^{\ k} - \frac{1}{2} R\delta_i^{\ k}\right) + \frac{1}{\sqrt{-g}}\,\partial_s \times \tag{6.5}$$

$$\left\{\sqrt{-g}\left[\frac{1}{2\varkappa}\left(g^{km}g^{js} - g^{kj}g^{sm}\right)\partial_j g_{im} + \delta_i^{\ s}\Omega^k - \delta_i^{\ k}\Omega^s + N_i^{\ ks}\right]\right\}$$

$$t_i^{\ ks} = \frac{1}{2\varkappa}\left(g^{kj}g^{sm} - g^{ks}g^{mj}\right)\partial_i g_{mj} + N_i^{\ ks}$$

$$\theta^{kij} = \frac{1}{4\varkappa}\left[\left(g^{ki}g^{ns} - g^{ks}g^{ni}\right)\Gamma^j_{sn} + \left(g^{kj}g^{ns} - g^{ks}g^{nj}\right)\Gamma^i_{ns}\right] -$$

$$\frac{1}{2}\Omega^k g^{ij} + \frac{1}{\sqrt{-g}}\,\partial_s\left(\sqrt{-g}\,N^{ksij}\right)$$

Components T^{ksij} in (6.4) are defined by the equality (6.3).

Since in virtue of Einstein's equations the mixed components of the pseudotensors of the gravitational field energy momentum, proposed by various authors, are represented in the form of antisymmetric components with three indices, hence it is possible to obtain as $t_i^{\ k}$ in (6.5) all of the proposed formulas for the energy-momentum pseudotensors with mixed indices even when $\Omega^i = 0$ and $\delta\Omega^i = 0$ merely by the choice of the arbitrary functions $N_i^{\ ks}$. For example, function $N_i^{\ ks} = 0$ corresponds to the Lorentz pseudotensor, and for the Einstein pseudotensor we have

$$N_i^{\ ks} = \frac{1}{2\varkappa}\left\{-\left(g^{km}g^{js} - g^{kj}g^{sm}\right)\partial_j g_{im} - \right.$$

$$\left.\frac{1}{\sqrt{-g}}\,g_{in}\partial_m\left[(-g)\left(g^{kn}g^{sm} - g^{sn}g^{km}\right)\right]\right\}$$

For $\Omega^i \neq 0$ the preceding results are also obtained by the overdetermination of $N_i^{\ ks}$. In particular, if in the Lagrangian (6.1) functions Ω^i are defined by the equality

$$\Omega^i = \frac{1}{2\varkappa}\left(g^{si}\Gamma_{sj}^{\ \ j} - g^{js}\Gamma_{js}^{\ \ i}\right) = \frac{1}{2\varkappa(-g)}\,\partial_s\left[(-g)\,g^{si}\right]$$

the Lagrangian Λ is independent of derivatives of Christoffel symbols, and formula (6.2) for δW becomes

$$\delta W = \int_\Sigma \left(t_i^{\ k}\delta x^i + N_i^{\ ks}\partial_s\delta x^i + \theta^{kij}\delta' g_{ij} + N^{ksij}\partial_s\delta' g_{ij}\right) n_k d\sigma \tag{6.6}$$

where $N_i^{\ ks} = -N_i^{\ sk}$ and $N^{ksij} = -N^{skij}$ are arbitrary, and for $t_i^{\ k}$ and θ^{kij} we have

$$t_i^{\ k} = \frac{1}{\varkappa}\left(R_i^{\ k} - \frac{1}{2} R\delta_i^{\ k}\right) - \tag{6.7}$$

$$\frac{1}{2\varkappa}\,\partial_s\left\{\frac{1}{\sqrt{-g}}\,g_{in}\partial_m\left[(-g)\left(g^{kn}g^{sm} - g^{sn}g^{km}\right)\right]\right\} +$$

$$\frac{1}{\sqrt{-g}}\,\partial_s\left(N_i^{\ ks}\sqrt{-g}\right)$$

$$\theta^{kij} = \frac{1}{4\varkappa}[(g^{ij}g^{sp} - g^{si}g^{jp} - g^{js}g^{ip})\Gamma^k_{sp} +$$
$$\left(- g^{ij}g^{sk} + g^{si}g^{jk} + g^{js}g^{ik})\Gamma^p_{sp}\right] + \frac{1}{\sqrt{-g}}\partial_s(\sqrt{-g}\, N^{ksij})$$

According to Einstein when $N_i^{ks} = 0$, components t_i^k in (6.7) are components of the pseudotensor of the gravitational field energy-momentum.

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